Mathematics and Computer Science for Modeling
Unit 5: Integration

Daniel Sabinasz
based on materials by Jan Tekülve and Daniel Sabinasz

Institut für Neuroinformatik, Ruhr-Universität Bochum

October 2, 2023

Dates

1. Mon 25.09. 15-17:30
2. Tue 26.09. 09:00-11:30, 15-17:30
3. Wed 27.09. 15-17:30
4. Thu 28.09. 15-17:30
5. Fri 29.09. 15-17:30
6. Mon 02.10. 09:00-11:30, 15-17:30
7. Wed 04.10. 15-17:30

Course Structure

Unit	Title	Topics
1	Intro to Programming in Python	*Variables, if Statements, Loops, Functions, Lists*
-	Full-Time Programming Session	*Deepen Programming Skills*
2	Functions in Math	*Function Types and Properties, Plotting Functions*
3	Linear Algebra	*Vectors, Trigonometry, Matrices*
4	Calculus	*Derivative Definition, Calculating Derivatives*

Course Structure

Unit	Title	Topics
5	Integration	*Geometrical Definition, Calculating Integrals*
6	Differential Equations	*Properties of Differential Equations*
-	04.10.23: Test	

Lecture Slides/Material

Use the following URL to access the lecture slides:

https://www.ini.rub.de/teaching/courses/preparatory_course_mathematics_and_computer_science_for_modeling_winter_term_2023

Overview

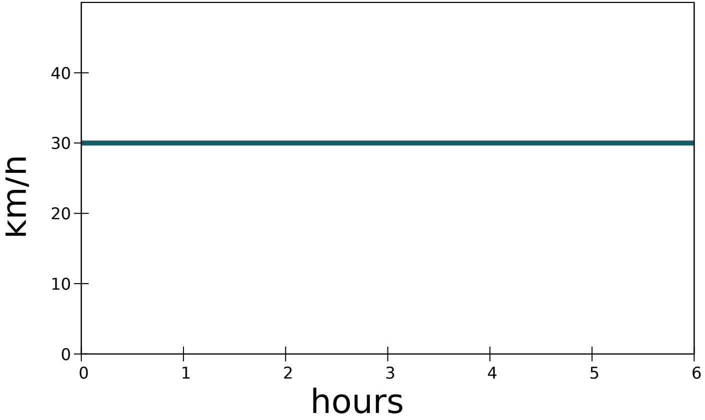
1. **Motivation**

2. **Mathematics**
   - Approximating the Area under a Curve
   - Calculating the Area under a curve
   - Improper Integrals

3. **Exercise**

From Velocity to Position

You drove 30 km/h for 6 hours. How far did you drive?



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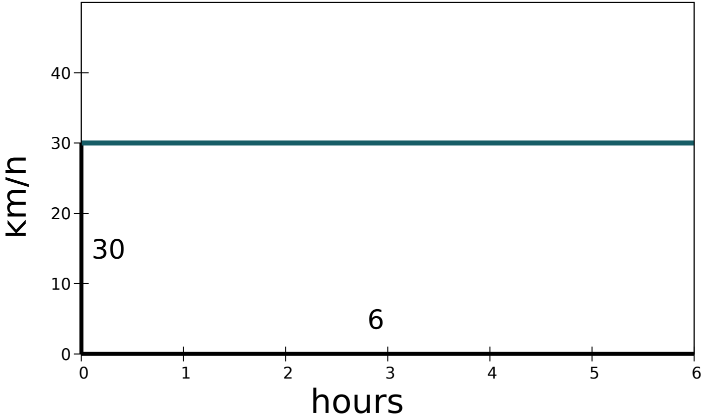

$$30\frac{\text{km}}{\text{h}} * 6\text{h} = 180\text{km}$$

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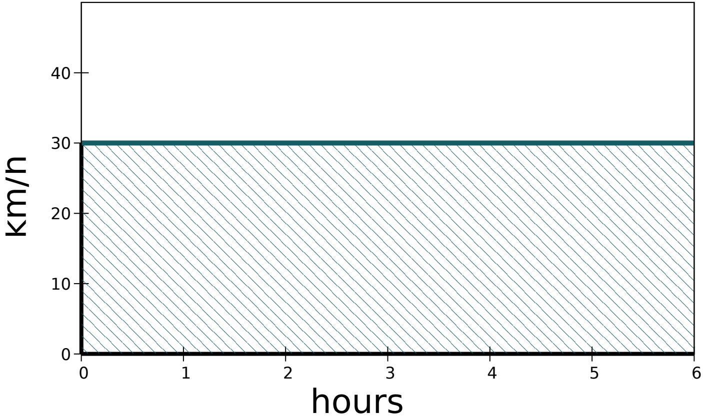

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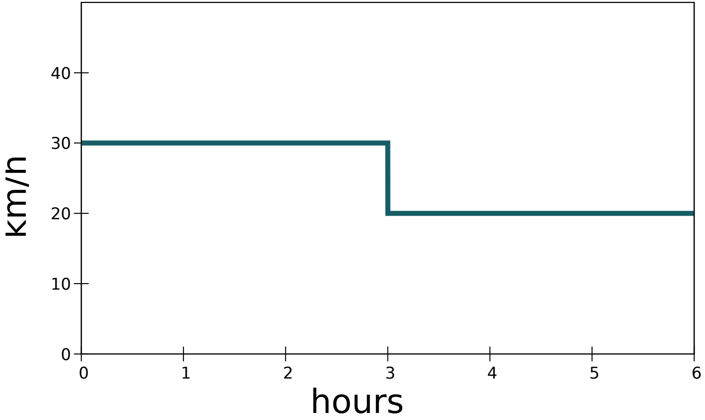
From Velocity to Position

Let's say you slowed down for the last 3 hours. How far did you get?



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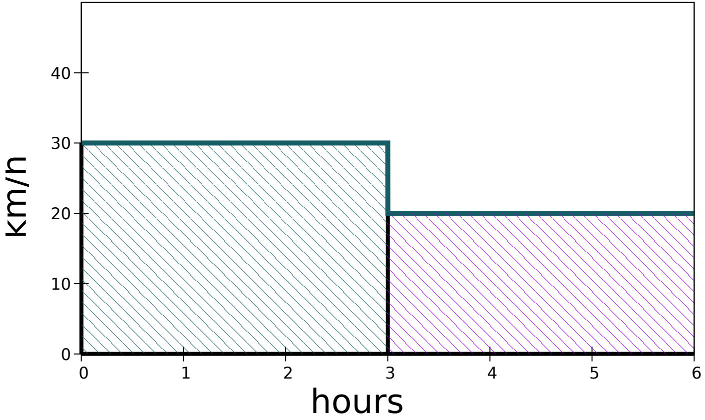
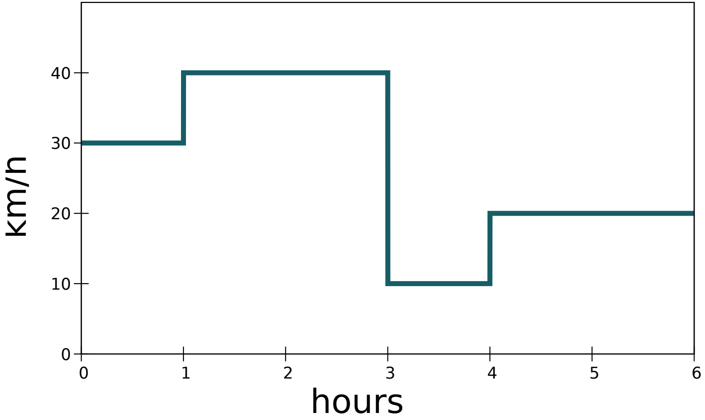


$$30\frac{\text{km}}{\text{h}} * 3\text{h} + 20\frac{\text{km}}{\text{h}} * 3\text{h} = 150\text{km}$$

From Velocity to Position

What if you mixed it up to not get bored?



From Velocity to Position

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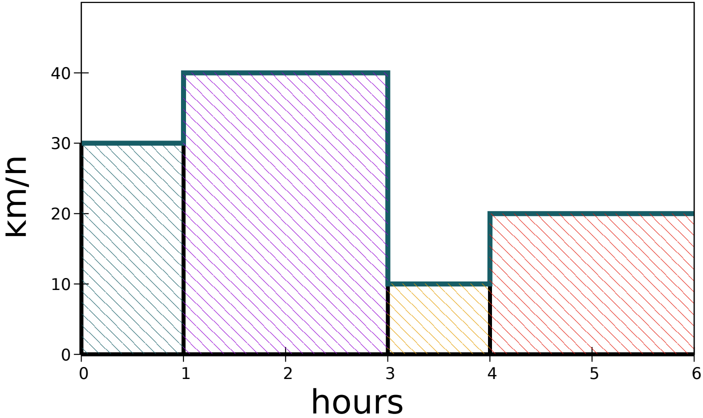
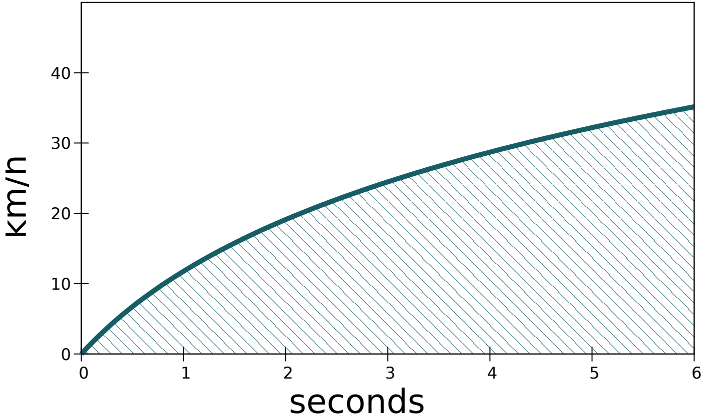


$$30\frac{\text{km}}{\text{h}} * 1\text{h} + 40\frac{\text{km}}{\text{h}} * 2\text{h} + 10\frac{\text{km}}{\text{h}} * 1\text{h} + 20\frac{\text{km}}{\text{h}} * 2\text{h} = 160\text{km}$$

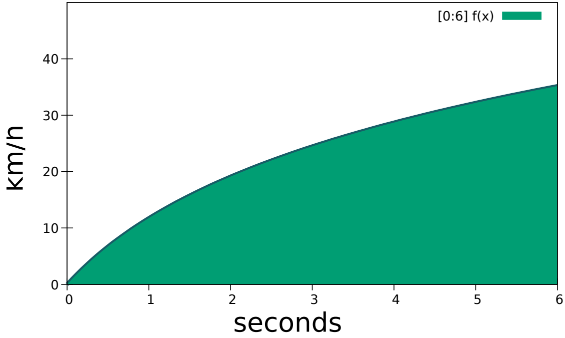
From Velocity to Position

But how about something realistic?



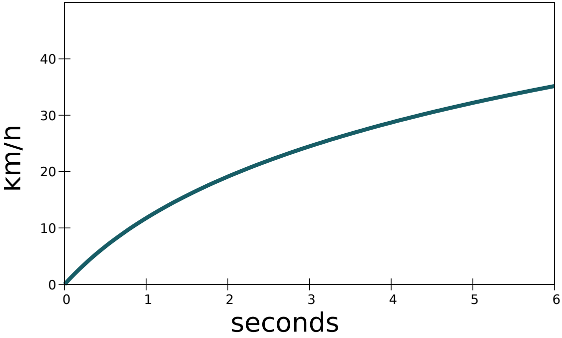
Approximation

- Not all areas can be calculated with rectangles



Approximation

- Not all areas can be calculated with rectangles
- One can however **approximate** them



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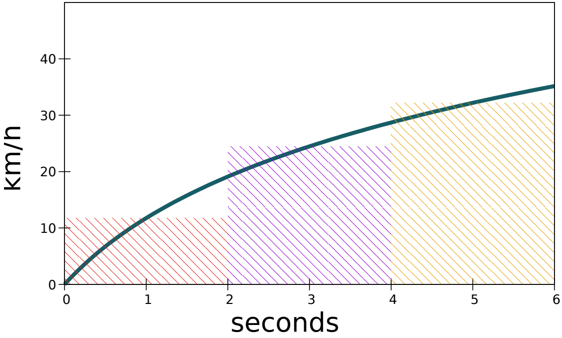

Approximation

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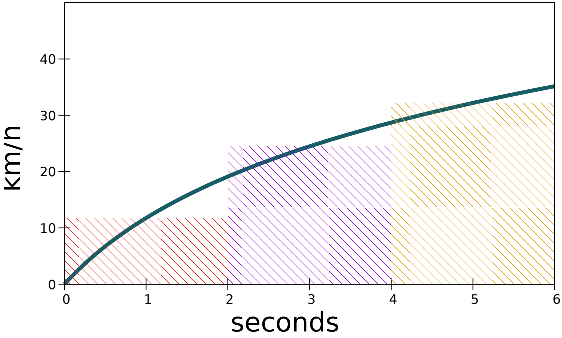
- The more rectangles the better the approximation becomes



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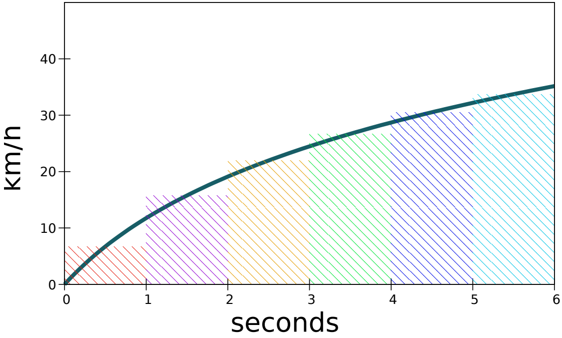


Midpoint Riemann Sum

Calculating Midpoints

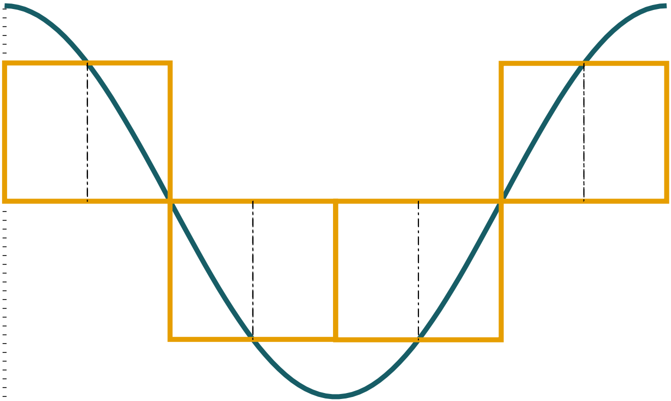
The **Midpoint Riemann Sum** is a way of approximating an integral with finite sums.
The are under the curve in a given interval $[x_i, x_{i+1}]$ can be approximated as the area of a rectangle with width $\Delta x = x_{i+1} - x_i$ and height $f(\frac{x_i+x_{i+1}}{2})$:

$$f(\frac{x_i + x_{i+1}}{2})\Delta x$$

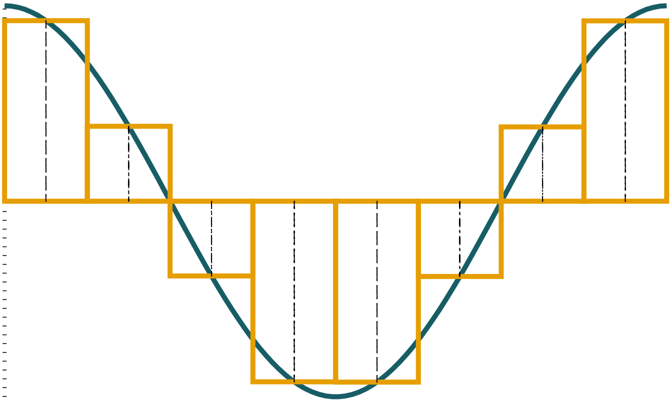
The sum over all intervals yields an estimation of the area under the curve

$$I_M = \sum_i^n f(\frac{x_i + x_{i+1}}{2})\Delta x$$

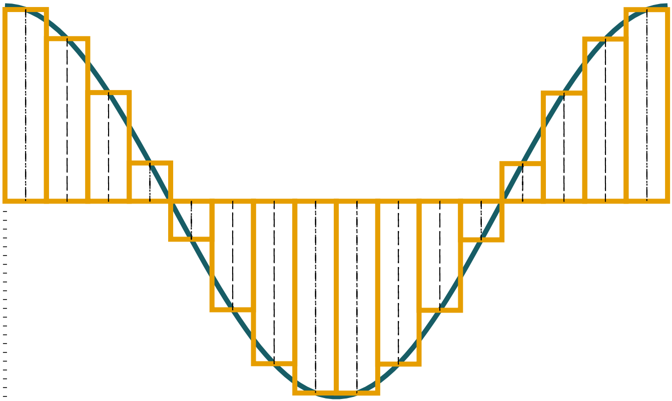
Midpoint Sums



Midpoint Sums



Midpoint Sums



From Sums to Integrals

Midpoint Sum: $f(\frac{x_i+x_{i+1}}{2})\Delta x$

The larger the number n of intervals, the smaller Δx and the better our approximation.

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What if n becomes infinitely large and Δx becomes infinitely small?

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Definite Integral

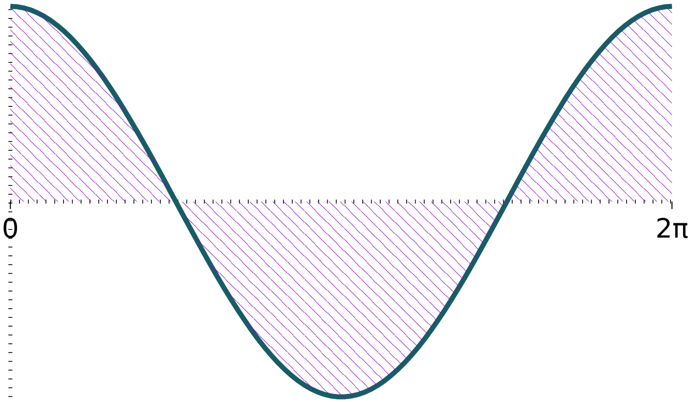
The **definite integral** of a function $f(x)$ between the **lower boundary** a and the **upper boundary** b

$$\int_a^b f(x)dx$$

is defined as the size of the area between f and the x-axis inside the boundaries. Areas above the x-axis are considered positively and areas below negatively.

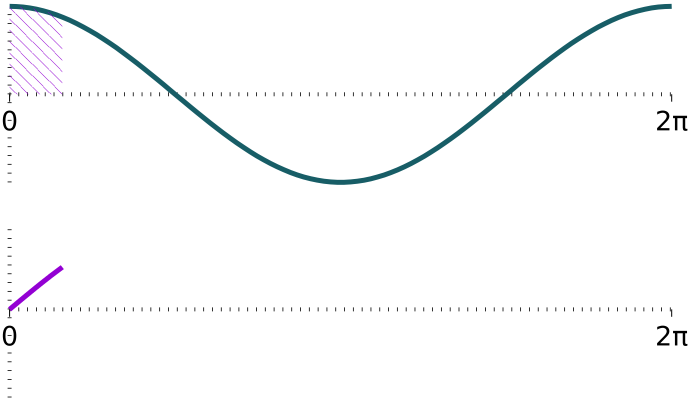
Definite Integral

$f(x) = cos(x)$ $\int_0^{2\pi} cos(x)dx$



Indefinite Integral

$f(x) = cos(x)$ $\int_0^x cos(x')dx'$



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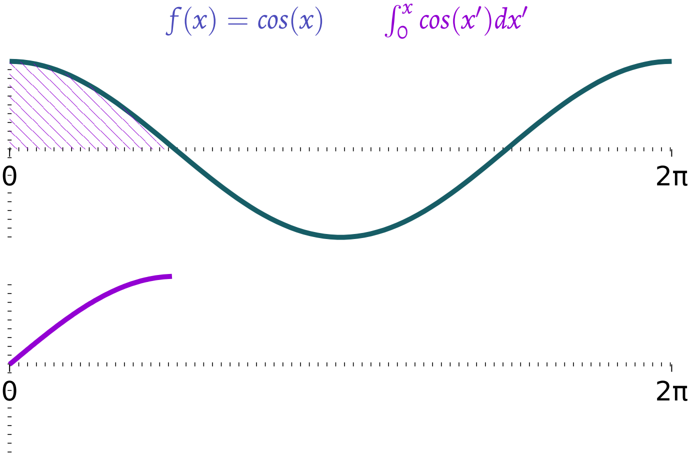

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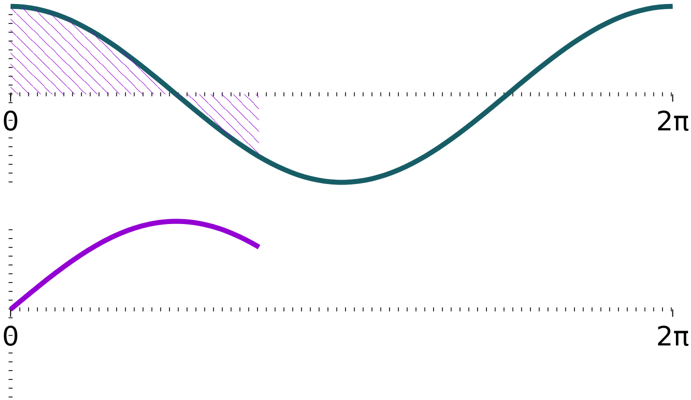

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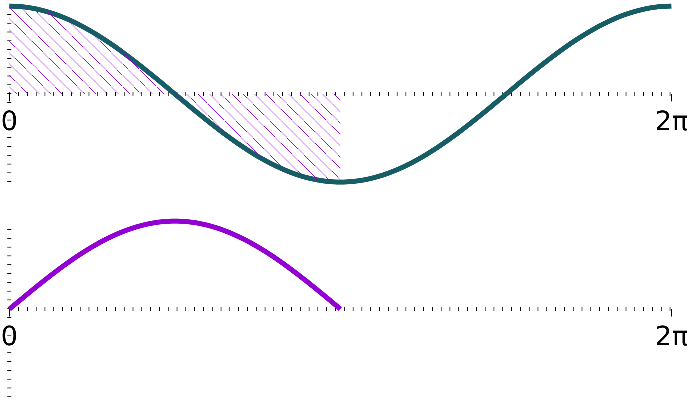

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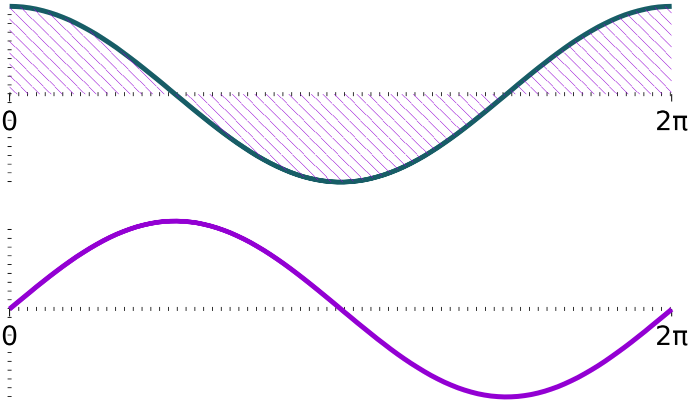
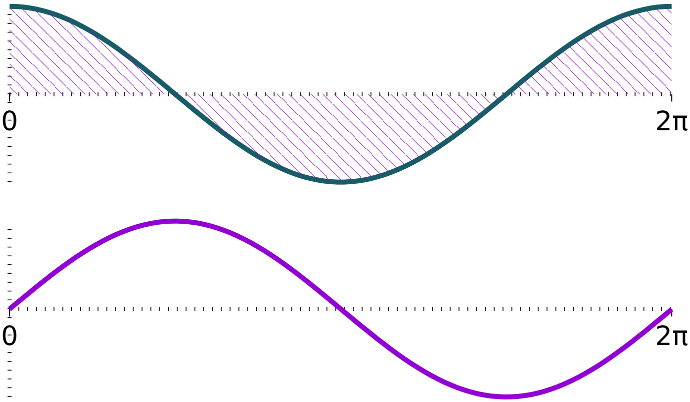


Indefinite Integral

$f(x) = cos(x)$ $\quad$ $\int_0^x cos(x')dx' = sin(x)$



Indefinite Integral

$$f(x) = cos(x) \qquad \int_0^x cos(x')dx' = \int cos(x')dx'$$

The Antiderivative

Definition

If f is a function with domain $[a, b] \to \mathbb{R}$ and there is a function F, which is differentiable in the interval $[a, b]$ with the property that

$$F'(x) = f(x),$$

then F is considered an **antiderivative** of f

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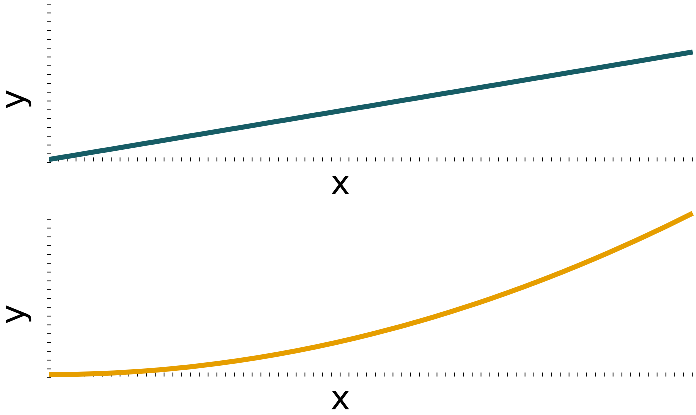
Properties of an antiderivative

- Differentiation removes constants, therefore $F(x) + c$ for any constant c is also an antiderivative
- Unlike with differentiation there are no fixed rules to compute an antiderivative from a given f

A function and its antiderivative

$f(x) = x$ $\quad$ $F(x) = \frac{1}{2}x^2$



The Fundamental Theorem of Calculus

First Fundamental Theorem of Calculus

One of the antiderivatives of a function can be obtained as the indefinite integral:

$$\int f(x')dx' = F(x)$$

- Intuition: The rate of change of the area under $f(x)$ is $f(x)$

The Fundamental Theorem of Calculus

Second Fundamental Theorem of Calculus

If f is integrable and continuous in $[a, b]$, then the following holds for each antiderivative F of f

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

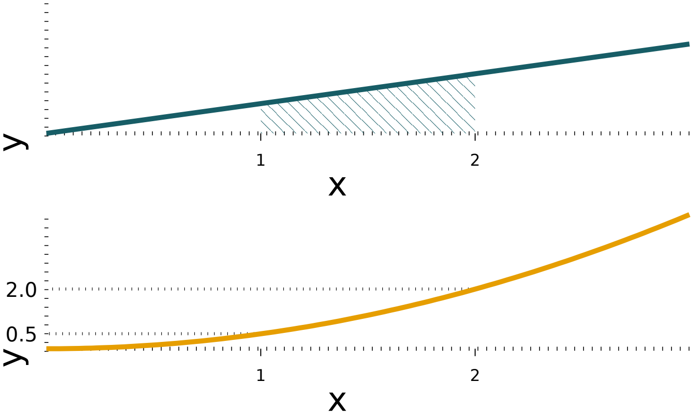
Example:

- Area under $f(x) = x$ between values 1 and 2

$$\int_1^2 xdx = \left[\frac{1}{2}x^2\right]_1^2 = \frac{1}{2}2^2 - \frac{1}{2}1^2 = 1.5$$

Definite Integral Example

$f(x) = x$ $\quad$ $F(x) = \frac{1}{2}x^2$ $\quad$ $\int_1^2 f(x)dx = F(2) - F(1)$



The Integral is a Linear Operator

Integration Rules

- **Summation**

$$\int_a^b f(x) + g(x) = \int_a^b f(x) + \int_a^b g(x)$$

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- **Scalar Multiplication**

$$\int_a^b cf(x) = c\int_a^b f(x)$$

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- **Boundary Transformations**

$$\int_a^b f(x) + \int_b^c f(x) = \int_a^c f(x) \qquad \int_a^b f(x) = -\int_b^a f(x)$$

Improper Integrals

Infinite Intervals

It is possible to calculate the area in infinitely large intervals. Intervals with an infinite boundary are called **Improper Integrals**

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

Example:

- Convergent improper integral

$$\int_1^{\infty} x^{-2}dx = \lim_{b\to\infty} \int_1^b x^{-2}dx = \lim_{b\to\infty} \left[-x^{-1}\right]_1^b = \lim_{b\to\infty} \left(-b^{-1} + 1\right) = 1$$

Exercise

Answer the following tasks using a piece of paper and a pocket calculator.

1. Given the Antiderivative $F(x) = 12x^2 + 5x$ of the function $f(x)$, calculate the area between $f(x)$ and the x-axis in the interval of $[-3, 5]$.

2. Calculate $\int_0^{\pi} \cos(x)dx$. Before applying the formula, look at a plot of $\cos(x)$. What kind of result would you expect?

Exercise Solutions

Exercise Solutions

1. The antiderivative is already given, therefore you only need to plug-in the beginning and the end of the interval.

$$\begin{aligned} [F(x)]_a^b =& F(b) - F(a) = F(5) - F(3) \\ =& 12 * 5^2 + 5 * 5 - \left(12 * (-3)^2 + 5 * (-3)\right) = 325 - 93 = 232 \end{aligned}$$

Exercise Solutions

1. The antiderivative is already given, therefore you only need to plug-in the beginning and the end of the interval.

$$\begin{aligned}[F(x)]_a^b =& F(b) - F(a) = F(5) - F(3) \\ =& 12 * 5^2 + 5 * 5 - \left(12 * (-3)^2 + 5 * (-3)\right) = 325 - 93 = 232\end{aligned}$$

2. Looking at the plot of $\cos(x)$ you can see that exactly the same area is enclosed above the x-axis as below the x-axis, therefore the total area has to be zero.
   To verify this analytically, you need to figure out the antiderivative of $\cos(x)$ first. From the lecture you know that $F(x) = sin(x)$.

$$[F(x)]_a^b = F(b) - F(a) = F(\pi) - F(0) = \sin(\pi) - \sin(0) = 0 - 0 = 0$$